Math 332H • March 13, 2009

Midterm Examination

This is a closed-book exam; neither notes nor calculators are allowed.

1) (30pts) For each, find **all distinct** values of *z*, and show roughly their locations as points in the complex plane:

(a) $z = \left(\sqrt{3} - i\right)^{1/4}$ (b) $z = \left(1 + i\right)^{i}$ (c) $\cos z = -4$.

2) (17pts) Sketch the mapping of the region $0 \le \operatorname{Re} z \le \pi$, $-1 \le \operatorname{Im} z \le 1$, under the transformation $w = exp(i\, z + 1)$

3) (17pts) Consider an analytic function $f(z) = f(x, y)$, where $z = x + i\, y$. Which of the following identities is/are true? Explain using the limit definition of the derivative.

(a) $\dfrac{df}{dz} = i\dfrac{\partial f}{\partial x}$ (b) $\dfrac{df}{dz} = -i\dfrac{\partial f}{\partial y}$ (c) $\dfrac{df}{dz} = \dfrac{\partial f}{\partial x}$ (d) $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y}$

4) (18pts) Calculate the following contour integrals (use any method you like):

(a) $\int_C |z|^2\, dz$, where *C* is a straight line from point (1, 4) to point (–1, 0).

(b) $\int_C \dfrac{z\, dz}{4 + z^2}$, where *C* is a unit quarter-circle in the 1^{st} quadrant, in the counter-clockwise direction, from point (1,0) to point (0,1).

5) (18pts) Which of these integrals equal zero for any closed contour *C* contained within the ring $1 \le |z| \le 2$? Explain your answer in terms of the Cauchy-Goursat and/or other theorems:

(a) $\oint_C z^{1/2} dz$ (b) $\oint_C \dfrac{dz}{3i + z}$ (c) $\oint_C \dfrac{dz}{(1 + 2z)^3}$

Alternative problems: you may do problem **(2B)** instead of **(2)**, and/or problem **(3B)** instead of **(3),** but note the smaller number of credit points:

2B) (14pts) Derive the expressions $u(x,y)$ and $v(x,y)$ for the real and imaginary parts of function $\cos z = \cos(x + i\, y)$, starting with the definition of cosine in terms of the exponential function, and using the Euler's formula.

3B) (14pts) Evaluate (using ε-δ definition), or prove that the limit does not exist:

(a) $\lim\limits_{z \to 0} \dfrac{\overline{z}^2}{z \cdot |z|}$ (b) $\lim\limits_{z \to 0} e^{-\frac{1}{|z|^2}}$